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Semiclassical Analysis of Low Lying Eigenvalues. IV. The Flea on the Elephant

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The two lowest eigenvalues $E_0(\lambda)$, $E_1(\lambda)$ of a symmetric double well tunnelling problem $-\Delta+\lambda^2 V$ as $\lambda\to\infty$ are considered and they are compared to the two lowest eigenvalues $\tilde{E}_0(\lambda)$, $\tilde{E}_1(\lambda)$ of $-\Delta+\lambda^2(V+W)$, where W is supported away from the well-bottoms of V. We determine the leading exponential splitting of various differences of the four numbers E_0, E_1, $\tilde{E}_0$, $\tilde{E}_1$. Related problems are discussed.

1. INTRODUCTION

This paper is a continuation of our series on the quasiclassical limit [7, 9, 10], especially on tunnelling problems [8, 9, 10]. One of our goals is to prove multidimensional analogs of the results of Jona-Lasinio, Martinelli, and Scoppola [5]. Our attention was drawn again to this paper by the recent work of Graffi, Grecchi, and Jona–Lasinio [2] who found a functional analytic proof of the 1-dimensional results of [5]. In this paper we will absorb some ideas of Helffer and Sjöstrand [3] into our framework; another of our goals will be to advertise their ideas. Independently and somewhat before this work, Helffer and Sjöstrand [4] also discussed multidimensional versions of [5, 2].

We will consider through most of the paper the two lowest eigenvalues $E_0(\lambda)$, $E_1(\lambda)$ (resp. $\tilde{E}_0(\lambda), \tilde{E}_1(\lambda)$) of $-\Delta+\lambda^2 V(x)=H(\lambda)$ (resp. $-\Delta+\lambda^2(V(x)+W(x))=\tilde{H}(\lambda)$), where

(a) V, W are C^∞ and nonnegative.

(b) $V(x)\geqslant\delta>0$ for $|x|>R$ for some R; W is bounded.

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(c) $V(Ax)=V(x)$ for a Euclidean transformation A of order 2.

(d) $V(x)=0$ if and only if $x=a$, b, where $Aa=b.\partial^2 V/\partial x_i\,\partial x_j$ (a) is nonsingular.

(e) $W(x)\equiv 0$ for x near a or b.

Thus W is a small flea on the elephant V. The flea does not change the shape of the elephant (in that we will see $\tilde{E}_1-\tilde{E}_0$ is exponentially small) but it can irritate the elephant enough so that it shifts its weight, i.e., we will see that the ground state, instead of being asymptotically in both wells, may reside asymptotically in only one well. These phenomena in one dimension are precisely what were discussed by Jona-Lasinio *et al.* [5].

We are interested in the quantities

$$a_{ij}=\lim_{\lambda\to\infty} -\lambda^{-1}\ln|E_i(\lambda)-\tilde{E}_j(\lambda)| \tag{1.1a}$$

$$\Delta=\lim_{\lambda\to\infty} -\lambda^{-1}\ln|E_1(\lambda)-E_0(\lambda)| \tag{1.1b}$$

$$\tilde{\Delta}=\lim_{\lambda\to\infty} -\lambda^{-1}\ln|\tilde{E}_1(\lambda)-\tilde{E}_0(\lambda)|. \tag{1.1c}$$

It will be useful to recall the results in [9] concerning Δ. For a non-negative function $f(x)$ we define the Agmon metric $\rho_f(x,y)$ by

$$\rho_f(x,y)=\inf\left(\int_0^1 \sqrt{f(\gamma(s))}\,|\dot{\gamma}(s)|\,ds\,|\,\gamma(0)=x,\gamma(1)=y\right)$$

$$=\inf\left(\int_0^t [\tfrac{1}{4}|\dot{\gamma}(s)|^2+f(\gamma(s))]\,ds\,|\,\gamma(0)=x,\gamma(T)=y\;0<T<\infty\right).$$

The equality of the two quantities is a result of Carmona and Simon [1]; we remark that ρ_f differs by $\sqrt{2}$ from that in [9], since we take $-\Delta$ here, where we took $-\frac{1}{2}\Delta$ there; ρ_V (resp. ρ_{V+W}) will be denoting ρ (resp. $\tilde{\rho}$).

The basic result in [9] is

THEOREM 1.1. ([9]) $\Delta=\rho(a,b)$.

This result was obtained by proving estimates on decay of eigenfunctions. Either of the proofs of decay in [9] shows

THEOREM 1.2. *If f is a function obeying hypotheses* (a), (b) *on V and f vanishes at some points, and if $\Omega_n(x,\lambda)$ is the* $(n+1)$th *eigenfunction of $-\Delta+\lambda^2 f(x)$, then*

$$\overline{\lim_{\lambda\to\infty}}\;\lambda^{-1}\ln|\Omega_n(x,\lambda)|\leqslant -\inf\{\rho_f(x,y)|\,f(y)=0\}. \tag{1.2}$$

The limit is uniform on compact sets, and for $|x|>2R$ (R given by hypothesis (b)),

$$|\Omega_n(x, \lambda)| \leqslant C_1 \exp(-C_2\lambda |x|) \tag{1.3}$$

for some $C_2>0$. In addition, if f runs through a compact set of functions obeying (a), (b) (uniform δ, R), then the convergence in (1.2) is uniform in f, and the compact sets and (1.3) hold uniformly in f.

To describe the lower bound estimates on Ω_0 in [9],

DEFINITION. We define the vacuum limiting set, $V(\Omega_0)$ by $x \in V(\Omega_0)$ if and only if for all neighborhoods N of x, $\lim_{\lambda\to\infty} \lambda^{-1} \ln \|\chi_N \Omega_0\| = 0$, where χ_N is the characteristic function of N. Thus, by Theorem 3.2, $V(\Omega_0) \subset \{x \mid f(x)=0\}$.

Either of the lower bound methods in [9] shows that

THEOREM 1.3. *Under the hypotheses of* Theorem 1.2:

$$\varliminf_{\lambda\to\infty} \lambda^{-1} \ln |\Omega_0(x, \lambda)| \geqslant -\inf\{\rho_f(x, y) \mid y \in V(\Omega_0)\}.$$

While neither of these theorems is explicitly stated in this generality in [9], the proofs of Theorems 2.1 and 2.3 of that paper prove Theorems 1.2 and 1.3 above. From Theorems 1.2, 1.3, we easily obtain estimates on $\tilde{E}_j - E_j$. Define

$$d_1 = \min(2\rho(a, \operatorname{supp} W), 2\rho(b, \operatorname{supp} W)) \tag{1.4a}$$

$$d_2 = \max(2\rho(a, \operatorname{supp} W), 2\rho(b, \operatorname{supp} W)) \tag{1.4b}$$

$$d = \rho(a, b). \tag{1.4c}$$

THEOREM 1.4.

(a) $\varlimsup_{\lambda\to\infty} \lambda^{-1} \ln[\tilde{E}_n(\lambda) - E_n(\lambda)] \leqslant -d_1$ (1.5a)

(b) $\varlimsup_{\lambda\to\infty} \lambda^{-1} \ln[\tilde{E}_0(\lambda) - E_0(\lambda)] \geqslant -d_2.$ (1.5b)

Proof. Let $\Omega_n(x\colon \lambda, \alpha)$ denote the nth eigenvalue of $-\Delta + \lambda^2(V+\alpha W)$. Then, by perturbation theory

$$\tilde{E}_n(\lambda) - E_n(\lambda)| = \lambda^2 \int_0^1 \left(\int W(x) |\Omega_n(x; \lambda, \alpha)|^2 \, dx\right) d\alpha. \tag{1.6}$$

By Theorem 1.2, we get an upper bound yielding (1.5a). (1.5b) comes from

(1.6) and the fact that for any ε there is an open set N_0 in supp W so that

$$\inf_{0 \leqslant \alpha \leqslant 1, x \in N_0} |\Omega_0(x; \lambda, \alpha)| \geqslant C_1 \exp(-\tfrac{1}{2}\lambda(d_2 + \varepsilon)).$$

This is just Theorem 1.3 (with a uniformity which is easy to check). ∎

In fact, we will see that $\lim \lambda^{-1} \ln|\tilde{E}_0(\lambda) - E_0(\lambda)|$ is never as small as $-d_2$ and often not as large as $-d_1$. Our goal in Sections 3–5 will be to treat the distinct cases:

Case 1 $(d < d_1 \leqslant d_2)$. $\Delta = \tilde{\Delta} = d$, $a_{00} = a_{11} = d_1$, $a_{01} = a_{10} = d$. Moreover, $V(\Omega_0(x, \alpha = 1))$ is $\{a, b\}$.

Case 2 $(d_1 < d \leqslant d_2)$. $\Delta = d$, $\tilde{\Delta} = d_1$, $a_{00} = a_{10} = d$, $a_{01} = a_{11} = d_1$. Moreover, $V(\Omega_0(X, 0 = 1))$ is a single point of a, b (the one with $\rho(\cdot$, supp $W) = d_2)$).

Case 3 $(d_1 < d_2 < d)$. $\Delta = d$, $\tilde{\Delta} = d_1$, $a_{00} = a_{10} = d_2$, $a_{01} = a_{11} = d_1$. Moreover, $V(\Omega_0(x, \alpha = 1))$ is a single point of a, b (the one with $\rho(\cdot$, supp $W) = d_2$).

There are two ways of summarizing and synthesizing these results. First, one can think of $\tilde{\Omega}_0$ as trying to minimize its energy subject to two rules: The vacuum can shift to only one well, but it costs an energy $O(e^{-\lambda d})$. If the vacuum is in both wells, the shift due to turning on W is $O(e^{-\lambda d_1})$, but if it is in the sole well with $\rho(\cdot$, supp $W) = d_2$, then $O(e^{-\lambda d_2})$. Thus, when $d_1 < d_2$, $\tilde{\Omega}_0$ is in one well if $d_1 < d$ and in both if $d < d_1$. This is the approach we will pursue in Sections 3–5, using as a preliminary an estimate of Helffer and Sjöstrand [3] proven in Section 2.

Another way is to let e_0, e_1 be the two eigenvalues of the matrix $\begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$ and $\tilde{e}_0$, $\tilde{e}_1$ the two eigenvalues of the matrix $\begin{pmatrix} \alpha_1 & \alpha \\ \alpha & \alpha_2 \end{pmatrix}$, where $\alpha_1 = \exp(-\lambda d_i)$, $\alpha = \exp(-\lambda d)$. Then the leading behavior of any difference of E's is identical to the leading behavior of the same pair of e's. This way of understanding the results will be explained in Section 6, using ideas of Helffer and Sjöstrand [3]. We will then apply these ideas to discuss some multiwell situations in Section 7.

As we were completing the preparation of this paper, we received a second paper of Helffer and Sjöstrand [4] which briefly analyzes this "flea on the elephant" situation. Their approach is like that in Section 6.

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2. Improved Decay of Eigenfunctions Following Helffer and Sjöstrand

Following ideas of Helffer and Sjöstrand [3], we prove

THEOREM 2.1. *Let V obey hypotheses* (a)–(d). *Let n be such that* $\lim E_j/\lambda = \lim E_n/\lambda$ *for exactly two values of* j. *Normalize* Ω_{2n+1} *relative to* Ω_{2n} *by* $\Omega_{2n+1}(a)/\Omega_{2n}(a) > 0$ (*see remark below*). *Then, for some* $\alpha_n(\lambda) \to 1$ *and* $\lambda \to \infty$:

$$\overline{\lim_{\lambda\to\infty}}\ \lambda^{-1} \ln |\Omega_{2n}(x;\lambda) + \alpha_n(\lambda)\, \Omega_{2n+1}(x;\lambda)|$$

$$\leqslant -\min(\rho(x,a), \rho(b,a)).$$

Remarks. (1) If $\Omega_{2n}(a)=0$, we can find b so that $\Omega_{2n}(a+\lambda^{-1/2}b)\neq 0$, in which case we normalize Ω_{2n+1} by $\Omega_{2n+1}(a+\lambda^{-1/2}b)/\Omega_{2n}(a+\lambda^{-1/2}b)>0$.

(2) Using ideas of [4], one can take $\alpha_n(\lambda)=1$.

(3) The correct (for $n=0$, the limit) $\overline{\lim}$ is $-\rho(x,a)$. This can be proven using an idea from [4], that the oprator P_n below has an integral kernel obeying $|P_n(x,y)| \leqslant C_\varepsilon e^{-(1-\varepsilon)\rho(x,y)}$.

Proof. Let $\eta_n^\delta(x,\lambda)$ and $E_n^\delta(\lambda)$ denote the $(n+1)$th eigenfunction and eigenvalue of $-\Delta+\lambda^2 V$ with Dirichlet boundary conditions on the sphere $|x-b|=\delta$. Then either of the methods of proof of Theorem 1.2 (in [9]) shows that

$$\overline{\lim_{\lambda\to\infty}}\ \frac{1}{\lambda} \ln |\eta_n^\delta| \leqslant -\rho(x,a) \tag{2.1}$$

(indeed $\rho(x,a)=\inf(\int_0^1 \sqrt{V(\gamma(s))}\,|\dot\gamma(s)|\,ds \mid \gamma(0)=x,\ \ \gamma(s)=a)$ can be replaced by the inf over all paths with $|\gamma(s)-b|\geqslant\delta$ for all s). Let j_δ be a C^∞ function which is 0 if $|x+b|\leqslant\frac{4\delta}{3}$ and 1 if $|x-b|\geqslant\frac{5}{3}\delta$ and let $\psi_n^\delta(x,\lambda)=j_\delta(x)\,\eta_n^\delta(x,\lambda)$. A simple estimate (including the fact that one can bound local L^2 norms of ∇f in terms of L^2 norms of f and ∇f; see Lemma C.2.1 of [6]) proves that for any k,

$$\|(H-E_n^\delta(\lambda))^k\, \psi_n^\delta\| \leqslant C_{k,\delta}\, e^{-\lambda d(\delta)} \tag{2.2}$$

$$d(\delta) = \inf(\rho(x,a) \mid |x-b| \leqslant 2\delta)$$

and

$$|1-\|\psi_n^\delta\|| \leqslant \tilde C_\delta\, e^{-2\lambda d(\delta)}. \tag{2.3}$$

The methods of [7] show that $|E_n^\delta - E_{2n}| = O(\lambda^l)$ for all l. Let P_n be the projection onto the span of Ω_{2n} and Ω_{2n+1}. Then

$$\|(1-P_n)f\| \leqslant C\lambda^{-1} \|(H-E_n^\delta)f\|. \tag{2.4}$$

Thus, (2.2) implies

$$\|(H-E_n^\delta(\lambda))^k (1-P_n)\psi_n^\delta\| \leqslant CC_{k+1,\delta}\lambda^{-1}e^{-\lambda d(\delta)}. \tag{2.5}$$

Since $V \geqslant 0$, $\|(H+1)^{-k}f\| \leqslant \|(-\Delta+1)^{-k}f\|$, so by a Sobolev estimate, (2.5) implies that for suitable M,

$$\|(1-P_n)\psi_n^\delta\|_\infty \leqslant D\lambda^{-1}e^{-\lambda d(\delta)}, \tag{2.6}$$

where $\|\cdot\|_\infty$ is the L^∞ norm. We can write

$$P_n\psi_n^\delta = a_n^\delta(\lambda)\,\Omega_{2n} + b_n^\delta(\lambda)\,\Omega_{2n+1}.$$

Using (2.6) at $x=b$ (or near b if Ω_{2n}, Ω_{2n+1} vanish there asymptotically), we find that

$$\alpha_n(\lambda) \equiv b_n^\delta(\lambda)/a_n^\delta(\lambda) \to 1.$$

By (2.3),

$$(a_n^\delta)^2 + (b_n^\delta)^2 - 1 = O(e^{-2\lambda d(\lambda)}).$$

Using the bounds (2.6) and (2.1), we find that

$$|\Omega_{2n} + \alpha_n\Omega_{2n+1}| \leqslant Ce^{-\lambda d(\delta)} + Ce^{-(1-\varepsilon)\lambda\rho(x,a)}.$$

Since $d(\delta) \to \rho(a,b)$, as $\delta \downarrow 0$, the theorem follows. ∎

3. The Case $d < d_1$

This case is easy. By Theorem 1.4, the shifts a_{00} and a_{11} are smaller than Δ, so $\tilde{E}_1 - \tilde{E}_0$ is comparable to $E_1 - E_0$, i.e., $\Delta = \tilde{\Delta}$. Moreover, we can use $\tilde{\Omega}_0$ as a trial function for H and find that

$$(\tilde{\Omega}_0, H\tilde{\Omega}_0) \leqslant (\tilde{\Omega}_0, \tilde{H}\tilde{\Omega}_0) = \tilde{E}_0 \ll E_1$$

in the sense that $\tilde{E}_0 - E_0/E_1 - E_0$ is exponentially small. It follows that

$$\tilde{\Omega}_0 = \Omega + O(e^{-\lambda(d_1-d)}) \tag{3.1}$$

and thus $V(\Omega_0(x, \alpha=1)) = \{a, b\}$. In fact, the argument shows that the analog of (3.1) holds for any $\Omega_0(x, \alpha)$ uniformly for $\alpha = 0, 1$. Thus, using (1.6), we immediately see that $a_{00} = d_1$. Once we know that $V(\Omega_0(x, \alpha = 1))$ is $\{a, b\}$, one can see that away from those x with $\rho(x, a) = \rho(x, b)$ one has $\lim_{\lambda \to \infty} -(1/\lambda) \ln |\Omega_1(x, \lambda)|^2 = \min(\rho(x, a), \rho(x, b))$, so from (1.6), we see that $a_{11} = d_1$ also. Since $|E_n - \tilde{E}_n|$ $(n = 0, 1)$ is so small relative to $E_1 - E_0$, we conclude that $a_{01} = a_{10} = d$.

4. The Case $d_1 < d_2$

This is more involved. First, label the points so ρ (supp $W, b) = d_1 < \rho$ (supp $W, a) = d_2$. Next, note that since the geodesic from x to supp W only hits supp W at the end point, we have, for any $\alpha \geqslant 0$,

$$\rho_V(x, \text{supp } W) = \rho_{V + \alpha W}(x, \text{supp } W). \tag{4.1}$$

(4.1) has an important consequence, given Theorem 1.3,

$$V(\tilde{\Omega}_0(x)) \ni b \Rightarrow \underline{\lim} \frac{1}{\lambda} \ln(\tilde{\Omega}_0, \lambda^2 W \tilde{\Omega}_0) \geqslant -d_1$$

so that

$$V(\tilde{\Omega}_0(x)) \ni b \Rightarrow \underline{\lim} \frac{1}{\lambda} \ln(\tilde{\Omega}_0, (\tilde{H} - E_0)\, \Omega_0) \geqslant -d_1.$$

Put differently,

$$V(\tilde{\Omega}_0(x)) \ni b \Rightarrow a_{00} \leqslant d_1. \tag{4.2}$$

Given this, we can use the bounds of Section 2 to show that $V(\tilde{\Omega}_0(x)) = \{\alpha\}$. For let $\psi = (\Omega_0 + \alpha\Omega_1)/\sqrt{1 + \alpha^2}$ with α given by Theorem 2.1. Then, by that theorem

$$\overline{\lim} -\frac{1}{\lambda} \ln(\psi, W\psi) \leqslant -\min(2d, d_2) \equiv d_3. \tag{4.3}$$

Since

$$(\psi, H\psi) = E_0 + \frac{\alpha^2}{1 + \alpha^2}(E_1 - E_0),$$

we conclude that

$$\begin{aligned}(\psi, \tilde{H}\psi) &\leqslant E_0 + \frac{\alpha^2}{1 + \alpha^2}(E_1 - E_0) + O(e^{-\lambda(1 - \varepsilon)d_3}) \\ &\leqslant E_0 + O(e^{-\lambda(1 - \varepsilon)d}).\end{aligned} \tag{4.4}$$

Given (4.2), we conclude

$$V(\tilde{\Omega}_0(x)) = \{a\}. \tag{4.5}$$

Given this, it is easy to see that $\|\tilde{\Omega}_0 - (1/\sqrt{2})(\Omega_0 + \Omega_1)\| \to 0$ and that

$$(\tilde{\Omega}_0, H\tilde{\Omega}_0) \geqslant \tfrac{1}{2}(E_0 + E_1).$$

The last result and (4.4) (and $\alpha \to 1$) show that

$$\tilde{E}_0 - E_0 = \tfrac{1}{2}(E_1 - E_0) + O(e^{-\lambda(1-\varepsilon)d_3})$$

and, in particular,

$$\lim_{\lambda \to \infty} \frac{\tilde{E}_0 - E_0}{E_1 - E_0} = \frac{1}{2}. \tag{4.6}$$

From (4.6) we immediately conclude that

$$a_{00} = a_{10} = d. \tag{4.7}$$

Let $\tilde{\psi}_a$ (resp. $\tilde{\psi}_b$) be a cutoff function j_a (resp. j_b) times the Dirichlet boundary condition ground states for $\tilde{H}$ with zero boundary conditions on $\{x \mid |x-b| = \delta\}$ (resp. $\{x \mid |x-a| = \delta\}$) for δ small. As in Section 2 (following Helffer and Sjöstrand [3]), $\tilde{\psi}_a$ and $\tilde{\psi}_b$ lie in the span of $\tilde{\Omega}_0$, $\tilde{\Omega}_1$ up to errors (in L^∞) of order $e^{-\lambda(1-\varepsilon)d}$ and thus, since $\tilde{\psi}_a$, $\tilde{\psi}_b$ are almost orthogonal, $\tilde{\Omega}_1$ lies in the span of $\tilde{\psi}_a$ and $\tilde{\psi}_b$. Since (4.5) holds, $\tilde{\Omega}_1$ must have a $\tilde{\psi}_b$ component, and thus, by Theorem 1.3 (or rather, its analog with Dirichlet boundary conditions),

$$\overline{\lim} \frac{1}{\lambda} \ln(\tilde{\Omega}_1, \lambda^2 W \tilde{\Omega}_1) \geqslant -d_1. \tag{4.8}$$

From (4.8) we conclude that $a_{11} = a_{01} = \tilde{\Delta} = d_1$. This completes the proof of all we said we would prove in this case. It should be possible to prove that

$$\lim_{\lambda \to \infty} -\frac{1}{\lambda} \ln\left[\tilde{E}_0 - \frac{1}{2}(E_0 + E_1)\right] = d_2$$

using the improvement to Theorem 2.1 indicated in Remark 3 after that theorem.

5. The Case $d_1 < d$

The analysis is quite similar to that in Section 4; (4.2) still holds, as does (4.4), so

$$V(\tilde{\Omega}_0(x)) = \{a\}. \tag{5.1}$$

Moreover, returning to (4.4) and using (4.3) and $d_2 < d < 2d$, we see that

$$\overline{\lim}\, \frac{1}{\lambda} \ln |\tilde{E}_0 - E_0| \leqslant -d_2.$$

Combining this with Theorem 1.4(b), we see that

$$a_{00} = a_{10} = d_2. \tag{5.2}$$

Given (5.1), our analysis in the last section leading to (4.8) is still valid, so

$$a_{11} = a_{01} = d_1$$

and this implies that

$$\tilde{\Delta} = d_1.$$

6. The Interaction Matrix of Helffer and Sjöstrand

It is probably worth describing very briefly the approach that Helffer and Sjöstrand use for these problems. This may serve the reader as a useful introduction to part of their papers [3, 4]. Let $\tilde{\psi}_a$, $\tilde{\psi}_b$ be the cutoff eigenvectors for Dirichlet problems described in Section 4. Let $\tilde{P}$ be the projection onto $\tilde{\Omega}_0$, $\tilde{\Omega}_1$ and define $\tilde{\phi}_a = \tilde{P}\tilde{\psi}_a$; $\tilde{\phi}_b = \tilde{P}\tilde{\psi}_b$. Let $\tilde{E}_a^D$, $\tilde{E}_b^D$ denote the eigenvalues for the Dirichlet problems defining $\tilde{\psi}_a$.

Define also the Dirichlet objects without tildes associated to H and note that, by symmetry, $E_a^D = E_b^D$. Introduce the matrices (without tildes also)

$$\tilde{E} = \begin{pmatrix} (\tilde{\phi}_a, H\tilde{\phi}_a) & (\tilde{\phi}_a, H\tilde{\phi}_b) \\ (\tilde{\phi}_b, H\tilde{\phi}_a) & (\tilde{\phi}_b, H\tilde{\phi}_b) \end{pmatrix}; \quad \tilde{N} = \begin{pmatrix} (\tilde{\phi}_a, \tilde{\phi}_a) & (\tilde{\phi}_a, \tilde{\phi}_b) \\ (\tilde{\phi}_a, \tilde{\phi}_b) & (\tilde{\phi}_b, \tilde{\phi}_b) \end{pmatrix}.$$

By standard linear algebra, $\tilde{E}_0$, $\tilde{E}_1$ are the eigenvalues of $(\tilde{N})^{-1/2}$, $(\tilde{\mathscr{E}}\tilde{N})^{-1/2}$ and the eigenvectors of this matrix determine now $\tilde{\Omega}_0$ as related to $\tilde{\phi}_a$, $\tilde{\phi}_b$.

By the Helffer and Sjöstrand ideas of Section 2,

$$\|\tilde{\phi}_a - \tilde{\psi}_a\| = O(e^{-(1-\varepsilon)\lambda d})$$

(where ε will depend on δ). Since $(\tilde{\psi}_a, \tilde{\psi}_b) = O(e^{-(1-\varepsilon)\lambda d})$

$$\tilde{N} = \mathbb{1} + O(e^{-(1-\varepsilon)\lambda d})$$
$$N = \mathbb{1} + O(e^{-(1-\varepsilon)\lambda d})$$

and a similar estimate holds for $N^{-1/2}$, $\tilde{N}^{-1/2}$.

By the noted symmetry $E_a^D = E_b^D$, one can see that

$$\mathscr{E} = E_a^D \mathbb{1} + \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

with $\lim_{\lambda \to \infty} -(1/\lambda) \ln \beta = d$ and from this, one immediately sees that $\Delta = d$. On the other hand,

$$\tilde{\mathscr{E}} - \mathscr{E} = \begin{pmatrix} x & z \\ z & y \end{pmatrix},$$

where we can use several facts:

(i)
$$\|\tilde{\phi}_a - \phi_a\| = O(e^{-(1-\varepsilon)\lambda d} + e^{-(1-\varepsilon)\lambda d_2})$$
$$\|\tilde{\phi}_b - \phi_b\| = O(e^{-(1-\varepsilon)\lambda d} + e^{-(1-\varepsilon)\lambda d_1}),$$

where we use our convention $\rho(a, \operatorname{supp} W) \geqslant \rho(b, \operatorname{supp} W)$; (i) follows from elementary perturbation theory and the estimate in Section 2.

(ii) $\langle \tilde{\phi}_a, \tilde{\phi}_b \rangle$, etc., are all $O(e^{-(1-\varepsilon)\lambda d})$.

(iii) $\lim_{\lambda \to \infty} (1/\lambda) \ln \tilde{E}_a^D - E_a^D = -d_2$; $\lim_{\lambda \to \infty} (1/\lambda) \ln \tilde{E}_b^D - E_b^D = -d_1$.

This follows from (1.6).

Thus, one can easily read off the various cases. For example, if $d_1 < d_2 \leqslant d$, we see that $z = O(e^{-\lambda(1-\varepsilon)d})$ $\lim (1/\lambda) \ln x = -d_2$, $\lim (1/\lambda) \ln y = -d_1$ and Case 3 can be deduced.

7. Some Remarks on Multiple Wells

In this final section, we discuss two examples of the Helffer–Sjöstrand philosophy [3, 4] which show its use in multiple well problems. The first involves a question left open in [9], and the second some examples in [5].

THEOREM 7.1. *Suppose that V is a nonnegative C^∞ function, with $V(x) \geqslant \delta > 0$ for $|x| > R$ (some R) so that V has only a finite number of zeros, each nondegenerate. Suppose that $a_1, \ldots, a_k$ are among these zeros, and that for some δ, the functions $V\chi_\alpha$ (where χ_α is the characteristic function of*

$\{x \mid |x-a_\alpha|<d\}$) are related to one another by Euclidean motions. Let e_0, $e_1,\ldots,$ denote the eigenvalues of $-\Delta+W$, where W is the quadratic approximation to V at some a_j. Then, for each j there are at least k eigenvalues $\{E_j^\alpha(\lambda)\}_{\alpha=1,\ldots,k}$ of $H(\lambda)-\ -\Delta+\lambda^2 V$ with

(1) $E_j^\alpha(\lambda)/\lambda \to e_j$

(2) $\overline{\lim}_{\lambda\to\infty}(1/\lambda)\lim |E_j^\alpha - E_j^{\alpha=1}|<0, \quad \alpha=2,\ldots,k.$

Remark. This says that *local* symmetry implies exponential splittings of eigenvalues.

Proof. Let η_j^1 be the $(j+1)$th eigenfunction of $-\Delta+\lambda^2 V\chi_\alpha$ with Dirichlet B.C. on the sphere $|x-a_\alpha|=\delta$ and define $\tilde{E}_j$ by $(-\Delta_D+\lambda^2 V\chi_\alpha)\,\eta_j^1$ so $\tilde{E}_j/\lambda\to e_j$. Let f be a C^∞ function which is supported in $\{x \mid |x|<\infty\}$, and which is 1 in $\{x \mid |x|<\frac{1}{2}\delta\}$. Let f^α be the images of f under those Euclidean motions which related the V_{χ_α}. Then $\phi_j^1=f\eta_j^1/\|f\eta_j^1\|$ are orthonormal and $\|(H(\lambda)-\tilde{E}_j)\,\phi_j^1\|$ are exponentially small. It follows that there are k eigenvalues all exponentially close to $\tilde{E}_j$. ▮

The second example involves a simple model of [5]. Take a potential V_0 on R (the method here is *not* 1-dimensional) obeying $V_0(x+1)=V_0(x)$ and $V_0(x)=V_0(-x)$ (this last is for simplicity exposition). Let W be a function obeying

(i) $W(x)=0$ if $\frac{1}{2}-\delta_0<x<L+\frac{1}{2}+\delta_0$

(ii) $W(x)>\varepsilon$ if $x<\delta_0$ or $x>L+1-\delta_0$,

where δ_0, $\varepsilon>0$ and let $V=V_0+W$ so V has L wells at $x=1,\ldots,L$. One can ask about the splittings of the eigenvalues of $H(\lambda)=-d^2/dx^2+\lambda^2 V$ and how the eigenstates are distributed between the wells. Let $d=\int_0^1\sqrt{V_0(y)}\,dy$ and $d(\delta)=\int_0^{1-\delta}\sqrt{V_0(y)}\,dy$.

Let M be the $L\times L$ matrix

$$M=\begin{pmatrix} 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 1 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and let $m_1\geqslant m_2\geqslant\cdots\geqslant m_L$ be its eigenvalues, and $\alpha^{(k)}$ its eigenvectors (i.e., $\sum M_{ij}\alpha_j^{(k)}=m_k\alpha_i^{(k)}$). Of course these are explicitly known, viz.,

$$m_k=\cos\left(\frac{\pi k}{L+1}\right), \qquad \alpha_j^{(k)}=\sqrt{\frac{2}{L}}\sin\left(\frac{\pi k j}{L+1}\right).$$

We introduce the symbol $\tilde{O}$ of Helffer and Sjöstrand: $\tilde{O}(e^{-a\lambda})$ means that for any ε, the quantity is $O(e^{-(a-\varepsilon)\lambda})$. In the theorem below, there are objects f, ψ. Then $\tilde{O}(e^{-a\lambda})$ means for any ε, we can choose f, ψ so that the errors are $O(e^{-(a-\varepsilon)\lambda})$.

We will describe the result for the case of the L lowest eigenvalue, but a very similar result (except we only have an upper bound on f_2 and we don't know if $f_2>0$) holds for excited clusters also. Let $E_j(\lambda)$, $j=0,\ldots,L-1$ denote the L lowest eigenvalue of $H(\lambda)$ and let $\zeta_j(x,\lambda)$ denote the corresponding eigenvectors.

THEOREM 7.2. *There exist functions $f_1(\lambda)$, $f_2(\lambda)$, and a function $\psi(x,\lambda)$ supported in x on $(-1,1)$ so that:*

(a) $f_1(\lambda)|\lambda \to$ constant; $f_2(\lambda)>0$; $\lim_{\lambda\to\infty}(1/\lambda)\ln|f_2(\lambda)| = -d$.

(b) *For* $|x|\geqslant\gamma>0$, $|\psi(x,\lambda)|\leqslant e^{-\Gamma(\gamma)\lambda}$; $\Gamma>0$, λ large.

(c) $E_j(\lambda)=f_1(\lambda)-m_{j+1}f_2(\lambda)+\tilde{O}(e^{-2\lambda d(\delta_0)})$.

(d) $\|\zeta_j(x,\lambda)-\sum_{i=1}^{L}\alpha_i^{(j)}\psi(x-i,\lambda)\|=\tilde{O}(e^{-d\lambda})$.

Remarks (1). (d) tells us about ratios like $\zeta_j(1,\lambda)/\zeta_j(2,\lambda)$.

(2) Since $d_0<\frac{1}{2}$ and V is even, $2d(\delta_0)>d$.

Proof. We first claim that one can replace $W(x)$ by any function of the same type controlling errors by the methods of the first part of the paper (here $d<d_1$! So the eigenfunctions will not change much. The eigenvalues will not change by more than $\tilde{O}(e^{-2\lambda d(\delta_0)})$). Hence, we ignore the effects of W except for the fact that it makes the bottom group of eigenvalues contain only L members.

Take δ small, and let $\eta(x,\lambda)$ and $f_0(\lambda)$ be the ground state of

$$\left[\frac{-d^2}{dx^2}+\lambda^2V_0(x)\right]\eta=f_0(\lambda)\,\eta;\qquad \eta(-1+\delta)=\eta(1-\delta)=0.$$

Let $\psi=j\eta$, where j is a function in C_0^∞ supported in $(-1+\frac{3}{2}\delta,1-\frac{3}{2}\delta)$, 1 on $(-1+2\delta,1-2\delta)$. Let P_λ be the projection onto $\{\zeta_j\}_1^L$, let $\psi_j(x,\lambda)=\psi(x-j,\lambda)$, $j=1,\ldots,L$ and let $\phi_j=P\psi_j$. Finally, let $f_1(\lambda)=(\psi_1,H\psi_1)$; $f_2^{\#}(\lambda)=(\psi_1,H\psi_2)$, $f_3(\lambda)=(\psi_1,\psi_2)$.

The first part of (a) and (b) are already known; (c), (d) will imply $f_2(\lambda)>0$ (since ζ_0 is positive) and the last result in (a) follows from standard tunnelling calculations. Thus, we need only prove (c), (d).

As in the last section, we are interested in $N^{-1/2}\mathscr{E}N^{-1/2}$, where N, $\mathscr{E}$ are the $L\times L$ matrices,

$$N_{ij}=(\phi_i,\phi_j);\qquad \mathscr{E}_{ij}=(\phi_i,H\phi_j).$$

We claim that (c), (d) follow easily from $(f_2 = f_2^\# - f_3 f_1)$.

$$N = \mathbb{1} - f_3(\lambda)\, M + \tilde{O}(e^{-2d\lambda}) \tag{7.1}$$

$$\mathscr{E} = f_1(\lambda)\, \mathbb{1} + f_2^\#(\lambda)\, M + \tilde{O}(e^{-2d\lambda}). \tag{7.2}$$

For (7.2), we note that if

$$\mathscr{E}_{ij}^\# = (\psi_i, H\psi_j)$$

then

$$\mathscr{E}_{ij}^\# = f_1(\lambda)\, \mathbb{1} + f_2(\lambda)\, M$$

so (7.2) is equivalent to

$$\mathscr{E}^\# - \mathscr{E} = \tilde{O}(e^{-2d\lambda}). \tag{7.3}$$

But (since P and H commute and $(1-P)^2 = (1-P)$),

$$\begin{aligned} |\mathscr{E}_{ij} - \mathscr{E}_{ij}^\#| &= |(P\psi_i, HP\psi_j) - (\psi_i, H\psi_j)| \\ &= |(H^{1/2}\psi_i, (1-P)\, H^{1/2}\psi_j)| \leqslant \|(1-P)\, H^{1/2}\psi_i\|\, \|(1-P)\, H^{1/2}\psi_j\| \\ &\leqslant \tfrac{1}{2}\, [\|(1-P)\, \psi_i\|^2 + \|(1-P)\, H\psi_j\|^2] \end{aligned}$$

and $\|(1-P)\,\psi_i\|$, $\|H(1-P)\,\psi_i\| = \tilde{O}(e^{-d\lambda})$ by the arguments in Section 2.

The proof of (7.1) is virtually identical to that of (7.2) if we note that $(\psi_i, \psi_i) = 1 + \tilde{O}(e^{-2d\lambda})$ given where j lives. ∎

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